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Periodic Inverse Problem for a New Hierarchy of Coupled Evolution Equations

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We study the inverse problem with periodic boundary condition for a new class of integrable nonlinear evolution equations. The multiphase periodic solutions for the nonlinear fields (p, q, r) are expressed in terms of the Riemann theta function, which is obtained via the linearization of the flows of the set of auxiliary variables "μ_j" on a Riemann surface. An explicit case is evaluated to obtain the form of the algebraic curve on which the variables "μ_j" move.

1. INTRODUCTION

The inverse problem of nonlinear equations falls into two categories—periodic and asymptotic (Bullough and Caudrey, 1980). While the latter class of problems has been studied in various situations, the former one has not been so widely investigated. The most exhaustively analyzed problem belonging to the periodic class is that of the Schrödinger operator or Sturm-Liouville type (Marchenko, 1974). The case of the sine-Gordon type equation was studied by Forest and McLaughlin (1982), and that of the Thirring model was analyzed by Date and Tanaka (1976). The corresponding situations in the case of derivative and mixed nonlinear Schrödinger equations were dealt with by Roy Chowdhury *et al.* (1985). Here we show that the same methodology with some modification can be used to study the periodic inverse problem for an extended class of coupled nonlinear equations, which contain the AKNS class as a special case.

2. FORMULATION

Since we intend to discuss a whole hierarchy of equations rather than a particular single equation, we start with the space part of the Lax equation,

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which reads (Roy Chowdhury and Roy, 1986)

$$\psi_x = \begin{pmatrix} \lambda - r & p \\ q & r - \lambda \end{pmatrix} \psi \tag{1}$$

When r becomes zero, (1) reduces to the AKNS problem. As in Forest and McLaughlin (1982), we construct the equations for the square eigenfunctions; $f = \psi_1^2$, $g = \psi_2^2$, $h = \psi_1\psi_2$:

$$\begin{aligned} f_x &= 2(-i\lambda + r)f + 2ph \\ g_x &= 2(i\lambda - r)g + 2qh \\ h_x &= qf + pg \end{aligned} \tag{2}$$

It is now easy to ascertain that $(h^2 - fg)_x = 0$. On the other hand, we write the time evolution of ψ as

$$\psi_t = \sum_{j=0}^{n} B_n \lambda^{n-j} \tag{3}$$

when B_n are 2×2 matrices of the form

$$\begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}$$

and the consistency of (1) and (3) yields the integrable hierarchy, whence we can obtain

$$\begin{aligned} f_t &= 2\sum (a_j f + b_j h)\lambda^{n-j} \\ g_t &= 2\sum (c_j h - a_j g)\lambda^{n-j} \\ h_t &= \sum (b_j g + c_j f)\lambda^{n-j} \end{aligned} \tag{4}$$

It is now assuring to note that without using any specific values of the coefficients a_j, b_j, and c_j we have

$$(h^2 - fg)_t = 0$$

Also, for completeness we quote from Roy Chowdhury and Roy (1986) the set of nonlinear equations generated:

$$\begin{pmatrix} p_t \\ q_t \\ r_t \end{pmatrix} = \begin{pmatrix} 2ib_{n+1} \\ -2ic_{n+1} \\ a_{nx} + pc_n - qb_n \end{pmatrix} \tag{5}$$

along with the recursion for the coefficients a_j, b_j, and c_j:

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \begin{pmatrix} iIr\,\partial/\partial x & i/2Iq\,\partial/\partial x & i/2Ip\,\partial/\partial x \\ ip & r - \frac{1}{2}\partial & 0 \\ iq & 0 & r + \frac{1}{2}\partial \end{pmatrix} \begin{pmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{pmatrix} \tag{6}$$

We now make the following ansatz for the solution of equations (2); we set

$$h=\sum_{n=0}^{N} h_n\lambda^{n+2}$$
$$g=\sum_{n=0}^{N} g_n\lambda^{n+1} \tag{7}$$
$$f=\sum_{n=0}^{N} f_n\lambda^{n+1}$$

along with

$$h^2-gf=P(\lambda)=\sum_{k=0}^{2N+2} P_k\lambda^{k+2}$$

so that the P_k are constants, with respect to both space and time variation. On the other hand, since g and f are polynomial of degree $N+1$, it is quite plausible to assume that they can be written in the form

$$g=g_N\prod_{j=1}^{N+1}[\lambda-\mu_j(x,t)]$$
$$f=f_N\prod_{j=1}^{N+1}[\lambda-\mu_j'(x,t)] \tag{8}$$

so that μ_j and μ_j' are the zeros of the analytic functions g and f. Substituting (8) in (2) and evaluating the resulting equation at $\lambda=\mu_k'$, we get (for f)

$$\frac{\partial\mu_k'}{\partial x}=-\frac{2i[P(\mu_k')]^{1/2}}{h_N\prod_{j=1;\,j\neq k}^{N+1}(\mu_k'-\mu_j')} \tag{9}$$

Similarly, for g we obtain

$$\frac{\partial\mu_k}{\partial x}=\frac{2i[P(\mu_k)]^{1/2}}{h_N\prod_{j=1;\,j\neq k}^{N+1}(\mu_k-\mu_j)} \tag{10}$$

In the derivation of (9) and (10) we have used the condition that $h^2|_{\lambda=\lambda_k}=P(\mu_k)$. On the other hand, using the series expansions (7) in equation (2), we obtain, by equating different powers of λ,

$$q=-\frac{ig_N}{h_N};\qquad p=+\frac{if_N}{h_N} \tag{11}$$

and

$$r=-i\sum\mu_j-\frac{iP_{2N+1}}{2P_{2N+2}}-\frac{1}{2}\frac{\partial}{\partial x}(\log g_N)$$

along with some useful identities:

$$\frac{h_{N-1}}{h_N}=\frac{P_{2N+1}}{2P_{2N+2}}; \qquad \frac{g_{N-1}}{g_N}=-\sum \mu_j$$
$$\frac{f_{N-1}}{f_N}=-\sum \mu_j' \tag{12}$$

Also, an expression for r can be written as

$$r=-i\sum \mu_j'-\frac{iP_{2N+1}}{2P_{2N+2}}+\frac{1}{2}\frac{\partial}{\partial x}(\log f_N) \tag{13}$$

Equations (11) and (13) are the basic equations for the inverse problem, so if we can ascertain the behaviors of $(g_N, h_N, f_N, \mu_j, \mu_j', s)$, then the nonlinear fields (p, q, r) are known. In the following we proceed to develop the evolution equations for these variables and show how they can be linearized on a Riemann surface. In the sequel we will refer to the set $(g_N, h_N, f_N, \mu_j, \mu_j')$ as auxiliary variables.

3. EVOLUTION OF THE AUXILIARY VARIABLES

The space and time evolution of the variables "μ_j" are now determined, from which the inverse periodic problem is solved via Abel's mapping. The space variation is already given in (9) and (10). For the required time evolution we consider a second-order flow in equation (3) instead of the general one. That is, we set $n=2$ in (3) and the coupled evolution equations that are generated are as follows:

$$q_t=\frac{q_{xx}}{2}+(qr_x)+iq_x+2rq_x-2q\int r_t\,dx+(+qr^2+2iqr-pq^2)$$
$$p_t=-\frac{p_{xx}}{2}+(pr_x)+ip_x+2rp_x+2p\int r_t\,dx+(-2pr^2-2ipr+p^2q) \tag{14}$$

These equations reduce to the usual nonlinear Schrödinger set if $r=0$. At this point we may note that even this new hierarchy of equations is Hamiltonian and can be written as

$$\begin{pmatrix} p_t \\ q_t \\ r_t \end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \partial/\partial x \end{pmatrix}\begin{pmatrix} \delta H/\delta p \\ \delta H/\delta q \\ \delta H/\delta r \end{pmatrix} \tag{15}$$

where H stands for a Hamiltonian. We now substitute the forms (7) and (8) in equation (4) corresponding to the case $n=3$, and after some simple algebraic manipulation deduce the following equations for μ_j, μ_j':

$$\frac{\partial \mu_{k'}}{\partial t} = -\frac{2[P(\mu_k')]}{h_N \prod_{j=1;\, j\neq k}^{N+1} (\mu_k' - \mu_j')} \left[-1 - \mu_k' + \sum_{j=1}^{N+1} \mu_j' + \frac{P_{2N+1}}{2P_{2N+2}} \right] \tag{16}$$

$$\frac{\partial \mu_k}{\partial t} = -\frac{2[P(\mu_k)]^{1/2}}{h_N \prod_{j\neq k;\, j=1}^{N+1} (\mu_k - \mu_j)} \left(1 + \mu_k - \sum_{j=1}^{N+1} \mu_j - \frac{P_{2N+1}}{2P_{2N+2}} \right) \tag{17}$$

and finally the equations for f_N and g_N.

4. EXPLICIT LINEARIZATION OF THE μ_j FLOW AND ABEL MAP

Due to the presence of the quantity $[P(\mu_k)]^{1/2}$ in equations (9), (10), and (16), each of these equations is to be defined properly on an n-sheeted Riemann surface, constructed by gluing such single sheets through the gap $[\mu_j, \mu_{j+1}]$. So these describe the motion of the auxiliary variables on a Riemann surface. Let us now construct the quantity

$$l_i(x,t) = -\sum_{l=1}^{N} C_{ji} \sum_{k=1}^{N} \int_0^{\mu_k} \frac{\lambda^{N-1}}{R(\lambda)}\, dx \tag{18}$$

where $R^2(\lambda) = \pi(\lambda - \lambda_j)$.

It is then easy to show that due to (9), (16), and (10), $l(x,t)$ evolves both with respect to space and time linearly, that is,

$$\frac{\partial l_i}{\partial x} = \text{const} = \alpha_i \quad \text{(say)}$$
$$\frac{\partial l_i}{\partial t} = \text{const} = \beta_i \quad \text{(say)} \tag{19}$$

whence we at once infer that

$$l_i(x,t) = \alpha_i x + \beta_i t + \gamma_i$$

So the motion is linearized via the change of variable (l_i). Now the whole problem is, given the l_i, how can one retrieve the old variables μ_j? It is actually the old Abel mapping and it is now convincingly proved that they are determined by multiphase Riemann Θ functions, with phases given by (l_i). The Riemann Θ function is defined as

$$\Theta(P,B) = \sum_k \exp[i\pi(BK,K) + 2\pi i(P,k)]$$

and

$$\theta(Z)=\theta(e-l(x,t))$$

$$l_i=\alpha_i x+\beta_i t+\gamma_i$$

$$l_i^0=\sum_{j=1}^{N} w_i(\mu_j(v,0))+\tfrac{1}{2}B_{ij}-j/2$$

whence B_{ij} denotes the periodic matrix given by $B_{\mu\nu}=\oint_{b\mu} d\omega_\nu$ and $d\omega_\nu$ are the differentials normalized according to

$$\oint_{a_\mu} d\omega_\nu=\delta_{\mu\nu} \tag{20}$$

a and b, respectively, denote the a cycles and b cycles defined on the Riemann surface. The differentials $d\omega_\nu$ are constructed according to

$$d\omega_\nu=\frac{C_{\nu 1}\lambda^{N-1}+\cdots+C_{\nu N}}{R(\lambda)},\qquad \nu=1,2,\ldots,N \tag{21}$$

5. DISCUSSION

Since it is known that the Riemann theta function is nothing but a generalization of the elliptic and hyperelliptic functions, they are also doubly periodic. Hence, once the μ_j are determined by them, the nonlinear variables (p, q, r) are bound to be periodic through equations (11). Lastly, since the application of the Riemann theta function is too abstract, one may have some idea about the μ_j variables through the following approach. Let us consider $n=1$ in equations (9) and (10), which then read

$$\begin{aligned}\frac{\partial\mu_1}{\partial x}&=\frac{2i[P(\mu_i)]^{1/2}}{\mu_1 h_2(\mu_1-\mu_2)}\\ \frac{\partial\mu_2}{\partial x}&=\frac{-2i[P(\mu_2)]^{1/2}}{\mu_2 h_2(\mu_1-\mu_2)}\end{aligned} \tag{22}$$

So we get

$$\frac{\partial\mu_1}{\partial x}\frac{\mu_1}{[P(\mu_1)]^{1/2}}+\frac{\partial\mu_2}{\partial x}\frac{\mu_2}{[P(\mu_1)]^{1/2}}=0$$

or

$$\frac{\partial}{\partial x}(\mu_1^2)\frac{1}{[P(\mu_1)]^{1/2}}+\frac{\partial}{\partial x}(\mu_2^2)\frac{1}{[P(\mu_2)]^{1/2}}=0 \tag{23}$$

The integration can now be facilitated if the odd P_i (which are all absolute constants) are zero; then we represent (Springer, 1960)

$$P(x)=(x^2-a^2)(x^2-b^2)(x^2-c^2)$$

and for $+\infty > \mu_1^2 > a^2$, $\infty > \mu_2^2 > a^2$, we get

$$S_n^{-1}\left[\frac{b^2(\mu_1^2-a^2)}{a^2(\mu_1^2-b^2)}\right]^{1/2}+S_n^{-1}\left[\frac{b^2(\mu_2^2-a^2)}{a^2(\mu_2^2-b^2)}\right]^{1/2}=0$$

We now utilize the following addition formulas for Sn functions

$$\mathrm{Sn}(u+v)=\frac{\mathrm{Sn}\, u\, \mathrm{Cn}\, v\, dn\, v+\mathrm{Cn}\, u\, dn\, u\, \mathrm{Sn}\, v}{1-K^2\, \mathrm{Sn}^2\, u\, \mathrm{Sn}^2\, v}$$

to deduce the equation of the algebraic curve on which (μ_1, μ_2) are constrained to move; we have

$$\phi(x, y)=0$$

where

$$\begin{aligned}\phi(x, y)=&\, y(x^2-a^2)^{1/2}(x^2-b^2)^{1/2}[a^2(y^2-b^2)-K^2b^2(y^2-a^2)]^{1/2}\\&+x(y^2-a^2)^{1/2}(y^2-b^2)^{1/2}\\&\times[a^2(x^2-b^2)-K^2b^2(x^2-a^2)]^{1/2}\end{aligned} \tag{24}$$

Equation (24), upon rationalization, gives the equation of an elliptic curve, the coordinates of which are usually parametrized in terms of elliptic functions. A similar analysis can also be done for the time variation of the auxiliary variables.

Lastly, it can be mentioned that our approach can yield the periodic solutions not only for a single equation, but for a whole hierarchy of equations, a point which was overlooked in previous publications.

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